

THE CASSON–GORDON INVARIANT AND LINK CONCORDANCE

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CASSON–GORDON invariants were developed in 1975 to prove the existence of algebraically slice knots in S^3 which are not smoothly slice, [1, 2]. Here we will develop a theory of Casson–Gordon invariants of smooth links which is applicable in high dimensions as well as in dimension 3. One main application will be to the study of concordances to boundary links, expanding on the work in [19]. We will give new proofs that many of the explicit examples of high dimensional links first studied by Cochran and Orr [4] are not concordant to boundary links.

Since their introduction, Casson–Gordon invariants have been used to address a variety of problems in knot theory. These include questions concerning double-null concordance in the classical dimension [11], higher dimensional double-null concordance [22], and amphicheirality and reversibility of knots [16, 18]. The study of applications to link concordance was initiated in [19]. Tim Cochran informs us that he is working on similar applications.

The question of which links are concordant to boundary links has a long history. In higher dimensions it is a pivotal problem in the classification of link concordance. Cochran and Orr [4] have recently announced significant new results in this area. In particular, they have described the first construction of higher dimensional links which are not concordant to boundary links. In dimension 3 the Milnor $\bar{\mu}$ -invariants provide obstructions to a link being concordant to a boundary link. Here Cochran and Orr produce the first examples of links for which the $\bar{\mu}$ -invariants vanish but which are not concordant to boundary links. In [19] it was shown that for a particular collection of the 3-dimensional examples in [4], Casson–Gordon invariants provide alternative obstructions to the existence of a concordance to a boundary link.

Throughout this paper we will work in the smooth category. Knots and links are always codimension 2. The paper is organized as follows. In Section 1 we briefly review the work of [19]. Section 2 presents a general definition of Casson–Gordon invariants for knots in S^{2n+1} , $n \geq 1$. In Section 3 a family of links taken from [4] is analyzed and is shown to contain no link concordant to a boundary link. The examples of this section are based on genus 1 knots in S^3 , and their higher dimensional analogues. Such knots have provided the basic examples for much of the work on knot concordance. For instance, they are central in all of [2, 4, 19, 22]. Calculable invariants are usually derived using 2-fold covers. When higher degree covers are needed, such as in [23], various *ad hoc* methods of computation have been developed. (See also [24].) In the course of the work of Section 3 we present

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a general algorithm for computing certain Casson–Gordon invariants, extending Gilmer’s work on genus 1 knots in S^3 to higher dimensions and to covers of degree greater than 2.

Section 4 returns to the study of classical link concordance. The work in [19] depended on quite restrictive conditions on the links being studied. Here a series of examples illustrate how those restrictions can be removed. Of particular interest is the construction of a family of two component links for which the explicit calculational methods of [4] fail to provide an obstruction. (A question that remains is whether or not these examples can be detected in the general Γ group studied by Cochran and Orr.)

The last two sections present general theories for Casson–Gordon invariants of links. In Section 5 we describe an algebraically defined group Ψ , and a function ψ from the set of 3-dimensional concordance classes of two component links to Ψ . It is shown that Ψ can be used to detect links which are not concordant to boundary links.

Section 6 defines Casson–Gordon invariants for high dimensional links, and proves that the invariants can detect links which are not concordant to boundary links.

§1. BACKGROUND

For the reader’s convenience we quickly review the argument in [19]. Suppose that a knot K_1 bounds a Seifert surface F_1 in the complement of a knot K_2 . If the link (K_1, K_2) is concordant to a boundary link $(K'_1, K'_2) = \partial(F'_1, F'_2)$ via a concordance (C_1, C_2) , then the surface $F_1 \cup C_1 \cup F'_1$ bounds a 3-manifold R_1 in $S^3 \times I$ which is disjoint from the surface $E = C_2 \cup F'_2$. If K_1 is slice, then so is K'_1 and K'_1 bounds a disk D in the 4-ball. Let R_2 be a 3-manifold bounded by $F'_1 \cup D$ in B^4 . By forming unions, we find that K_1 bounds a slice disk D' in the 4-ball such that $F_1 \cup D'$ bounds a 3-manifold R which is disjoint from a surface E bounded by K_2 .

If F_1 is genus 1, Gilmer [9] proved that there is a nontrivial simple closed curve γ on F_1 which represents torsion in $H_1(R)$, and that certain p -signatures of γ must vanish. (These signatures represent particular Casson–Gordon invariants.) The Seifert form necessarily vanishes on γ . In addition, the construction above shows that γ must link K_2 algebraically 0 times, since it represents torsion in $H_1(B^4 - E)$.

Let L be the link illustrated in Fig. 1, with J_2 unknotted. The bands on F_1 are untwisted, so that the Seifert form vanishes on the classes represented by x and y . In this case K_1 is slice and the above discussion applies. The only possible candidate for γ is the core of the left hand band. It has the knot type of J_1 , and hence the signature function of J_1 provides obstructions to the link being concordant to a boundary link.

§2. CASSON–GORDON INVARIANTS OF HIGH DIMENSIONAL KNOTS

Definitions. We begin by recalling earlier work in this area. If K is a knot in S^3 with 2-fold branched cover M , for each character χ of $H_1(M)$ to a finite cyclic group of prime

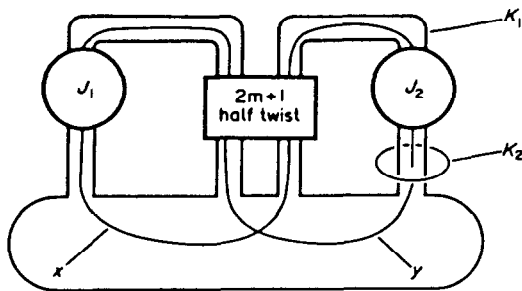


Fig. 1

power order Casson and Gordon defined two invariants, $\sigma(K, \chi)$ and $\tau(K, \chi)$. In [1] they initially showed that σ provides an obstruction to K being ribbon, while τ provides obstructions to K being slice. They then derived a relationship between σ and τ which yields a method using σ to prove knots are not slice. Later, Gilmer [10] expanded on [1] and interpreted σ (and also τ for genus one knots) in terms of certain classical knot signatures.

Ruberman [22] described how to generalize σ to higher dimensional knots, and showed that it provides obstructions to finding double null concordances. He also showed how to compute σ in terms of knot signatures in this higher dimensional setting. In some sense, the double null concordance problem is similar to the ribbon problem. In order to carry out our program for showing links are not concordant to boundary links we need to extend Ruberman's work to show that τ is well defined in the high dimensional setting, and that it can be computed in terms of classical signatures.

Overall the procedure is similar to that of [1] (see also [12]); we will only focus on the points where some modification of that work is required. We will not restrict to 2-fold covers, but will work with general q -fold covers, where q is a prime power.

The invariant τ can be defined as follows. Let K be a knot in S^{2n+1} (diffeomorphic to S^{2n-1}), $n \geq 1$, and let M be the q -fold branched cover of S^{2n+1} branched over K . Let F be a Seifert manifold for K capped off with a $2n$ -ball. Finally, let M_0 be the disjoint union of surgery on the lift of K to M with $-(F \times S^1)$. In the case $n = 1$ the surgery should have 0-framing. (It will soon be seen that in the case that K is algebraically slice, or $n = 1$, the component $F \times S^1$ need not be added.)

If χ is character from $H_1(M)$ to Z_d then one defines a character χ' from $H_1(M_0)$ to $Z_d \times Z$ as follows: χ restricts to a homomorphism ρ_1 from $H_1(M - K)$ to Z_d and since $H_1(M - K, Q) = Q$, there is a unique homomorphism ρ_2 from $H_1(M - K)$ to Z taking value 1 on a positive meridian of K . Together, ρ_1 and ρ_2 define a homomorphism $\rho_1 \times \rho_2$ mapping to $Z_d \times Z$ which extends uniquely to surgery on K . On the surgered component of M_0 , χ' is given by that extension. On $H_1(F \times S^1)$, χ' maps to the Z summand and is given by the map induced on homology by the projection onto S^1 .

We will show that for some integer s , which is odd if d is odd, the induced character defined on sM_0 extends over some $2n + 2$ manifold V . Using the $Z_d \times Z$ cover of V one defines a twisted homology group $H_{n+1}(V; Q(\zeta_d)(t))$ with coefficients in $Q(\zeta_d)(t)$, the field of rational functions over the cyclotomic field (see [2] p. 183). Moreover there is a Hermitian (or skew-hermitian if n is even) intersection pairing on $H_{n+1}(V; Q(\zeta_d)(t))$ with values in $Q(\zeta_d)(t)$. The difference of the Witt class of this pairing and that of the intersection pairing on $H_{n+1}(V)$ is an element $\{V, \chi\} \in L_\varepsilon(Q(\zeta_d)(t))$, where ε is 0 or 2 and $\varepsilon = 2n + 2 \pmod{4}$. We define $\tau(K, \chi)$ to be $(\{V, \chi\} - \{V, 0\})/s$ in $L_\varepsilon(Q(\zeta_d)(t)) \otimes Z_{(2)}$, or $L_\varepsilon(Q(\zeta_d)(t)) \otimes Q$ if d is even. As the rational case is easier, we proceed in the case of d odd.

To see that such an s and V exist, we begin by noting that the obstruction to finding s and V is represented by the element $[M_0, \chi]$ in $\Omega_{2n+1}(K(Z_d \times Z)) \otimes Z_{(2)}$. According to [6, Section 44] $\Omega_{2n+1}(K(Z_d \times Z)) \otimes Z_{(2)} \cong \Omega_{2n+1}(K(Z_d)) \otimes Z_{(2)} \oplus \Omega_{2n}(K(Z_d)) \otimes Z_{(2)}$. We next note that the inclusion of $\Omega_* \otimes Z_{(2)}$ into $\Omega_*(K(Z_d)) \otimes Z_{(2)}$ is an isomorphism. This is true with bordism replaced by homology via a transfer argument. It follows for bordism groups using the bordism spectral sequence.

Since M bounds a branched cover of B^{2n+2} over a pushed in Seifert manifold for K , and $F \times S^1$ bounds $F \times B^2$, it follows that M_0 is trivial in Ω_{2n+1} . The image of M_0 in $\Omega_{2n} \otimes Z_{(2)}$ is represented by a Seifert manifold for K , capped off with a ball, union $-F \times \{\text{pt.}\}$. (Again, see [6], Section 44.) Any two capped off Seifert manifolds for a knot are cobordant and hence this class is trivial as well. The cobordism between Seifert manifolds is constructed by taking disjoint lifts of the Seifert manifolds in the infinite cyclic cover of surgery on the knot.

THEOREM 1. *The value of $\tau(K, \chi)$ is independent of the choice of F and V .*

Proof. We first note that $\tau(K, \chi)$ is independent of the choice of F . If F' is a second Seifert manifold, then $(F \times S^1)$ and $-(F' \times S^1)$ cobound a manifold $N^{2n+1} \times S^1$ over Z . Let M'_0 denote M_0 with $(F \times S^1)$ replaced by $(F' \times S^1)$. If V is used to compute $\tau(K, \chi)$ when F is used, let $V' = V \cup s(N^{2n+1} \times S^1)$. The boundary of V' is some multiple of M'_0 . The value of τ that is computed using V' is the same as that for V with an added term coming from $N^{2n+1} \times S^1$. We claim that the added factor is trivial. To see this, note that the intersection form on $N^{2n+1} \times S^1$ is Witt trivial, as the manifold bounds $N^{2n+1} \times B^2$. On the infinite cyclic cover all intersections in H_{n+1} are trivial as the cover is diffeomorphic to $N^{2n+1} \times R^1$.

It must also be shown that different choices of V , say V_1 and V_2 , yield the same value for τ . As in the classical case, one needs to show that $\{W, \chi'\}$ vanishes for the $2n+2$ manifold W formed as the boundary union of appropriate multiples V_1 and V_2 , where χ' is the character induced from those coming from V_1 and V_2 . To do this one would like to show that W (or some multiple of W) bounds over $Z_d \times Z$. Unfortunately, in this case the obstruction is in $\Omega_{2n+2}(K(Z_d \times Z)) \otimes Z_{(2)}$, and it may happen that W is nontrivial here.

However, after again taking odd multiples, the above bordism argument implies that W is bordant to a class in the image of $\Omega_{2n+2}(Z) = \Omega_{2n+2} \oplus \Omega_{2n+1}$. By [6] such classes are represented by $2n+2$ manifolds with trivial representations to Z , and the products of S^1 with $2n+1$ manifolds with the representations to Z factoring through projection onto the S^1 . In the first case the forms on the cover and the base are the same, and so the difference of the Witt classes is zero. In the second case, the cover is the homotopy type of a $2n+1$ manifold and hence the form is trivial. The base is the product with S^1 , and hence bounds. Therefore its form is Witt trivial as well. $\square$

Comments. In the above discussion if d were even it would be possible to use coefficients from $Z[1/d]$ rather than to go to Q . However, it can be shown that $L_e(Q(\zeta_d)(t))$ contains only 2 torsion [16], and nothing is gained by this approach. Litherland [17] has constructed examples, with $n=1$, for which the use of $Z_{(2)}$ coefficients yields stronger results than are available working over the rationals. In Section 4 we will mention further examples of a different type.

Observe that rather than using $F \times S^1$ above, we could use any F' cobordant to F . In particular, as a Seifert manifold for an algebraically slice knot is null cobordant, the $F \times S^1$ factor is not needed if K is algebraically slice. Secondly, note that there is an alternative approach to defining τ . Rather than perform surgery on the lift of K to M and taking the disjoint union with $-F \times S^1$, one can replace a tubular neighborhood of K with $-F' \times S^1$, where F' is any Seifert manifold for K . This approach applies to the case of knots consisting of homotopy spheres. The proof of the above theorem is slightly more complicated in this setting. In the case K is diffeomorphic to S^{2n-1} the two approaches yield the same invariant.

In the classical dimension, Casson and Gordon proved that if K is slice and $H_1(M)$ is nonzero then there are some characters which extend over the q -fold branched cyclic cover of B^4 along the slice disk. Then the unbranched cover of the exterior of the slice disk is a 4-manifold (V in the above discussion) which can be used to calculate τ . If the original character has prime power order they show $\tau = 0$. In the higher dimensional setting it can transpire that no character on $H_1(M)$ extends to the q -fold branched cyclic cover of B^{2n+2} along the slice disk, and for this reason the Casson–Gordon invariants do not provide obstructions to slicing algebraically slice knots. However, if for some reason it is known that some character does extend, then their proof goes through to show that the corresponding τ is zero. Sometimes the hypothesis that a link of two components is concordant to

a boundary link allows us to conclude that a character on the branched cyclic cover along one of the components must extend as above. This yields an obstruction to being concordant to a boundary link.

§3. HIGH DIMENSIONAL LINKS NOT CONCORDANT TO BOUNDARY LINKS.

Cochran and Orr described a family of links, L_m , constructed as follows. Let K be a knot in S^{2n+1} . Let F be a untwisted trivial I bundle over K and attach a $2n$ -dimensional 1-handle to F . One end of the 1-handle is attached to each end of F . The core of the 1-handle along with an arc in $F \times I$ forms an embedded S^1 on F which should link K algebraically m times when pushed off of F in the positive direction. More specifically, that pushed off S^1 should be homotopic to m times the meridian of K . The negative push off is homotopic to $m + 1$ meridians. The resulting $2n$ -manifold, F_1 , has boundary K_1 . The other component of the link consists of a linking $(2n - 1)$ -sphere to the 1-handle. Figure 1 can be viewed as a schematic diagram of this link, with J_1 corresponding to K , and J_2 trivial.

We will show that if L_m is concordant to a boundary link, then certain Casson-Gordon invariants of K_1 will vanish. Then, generalizing results in a variety of papers, including [16, 22, 23 and 24], we will calculate these Casson-Gordon invariants in terms of invariants of J_1 . Before going into the details, we need a fairly precise description of the homology of the q -fold branched cover of S^{2n+1} branched over K_1 .

Let M_q denote the q -fold branched cover of S^{2n+1} branched over K_1 . The Seifert manifold F_1 for K_1 constructed above is built with a 0-handle, a $2n$ -handle and a 1-handle. Let S be a linking S^1 to the $2n$ -handle, and let T be a linking $(2n - 1)$ -sphere to the 1-handle. In Fig. 1, S and T could be illustrated schematically by linking circles to the two bands in the Seifert surface. The lifts of S and T to M_q will be denoted $\{S_i\}_{i=1, \dots, q}$ and $\{T_i\}_{i=1, \dots, q}$, with each set cyclically permuted by the Z_q action. The following Proposition is proved in Appendix 1.

PROPOSITION 2. *If $n = 1$, $H_1(M_q) = Z_a \times Z_a$, where $a = ((m + 1)^q - m^q)$. If $n > 1$, $H_1(M_q) = Z_a$, $H_{2n-1}(M_q) = Z_a$, and all other homology groups are trivial other than in dimensions 0 and $2n + 1$. For $n = 1$, the (nonsingular) linking form on $H_1(M_q)$ with values in Q/Z is hyperbolic with respect to the subgroups of $H_1(M_q)$ generated by S_1 and T_1 . In dimensions greater than 3 the linking form pairs these generators of H_1 and H_{2n-1} nontrivially. In $H_1(M_q)$, $S_i = m^*(m + 1)S_{i+1}$, $i < q$, and $S_q = m^*(m + 1)S_1$, where m^* is the multiplicative inverse of $m \bmod(a)$.*

THEOREM 3. *If L_m is concordant to a boundary link, then for all prime powers q , and for all χ of prime power order defined on $H_1(M_q)$ (which vanish on T_1 if $n = 1$), $\tau(K, \chi)$ is zero.*

Proof. The argument in [19] applies in all dimensions and shows that if the link L_m is concordant to a boundary link, then K_1 bounds a slice disk, D , which together with F_1 bounds a $(2n + 1)$ -manifold, R , disjoint from a $(2n)$ -manifold, E , bounded by K_2 . Let V denote the q -fold branched cover of B^{2n+1} branched over D . As E is disjoint from R , it lifts to V , and hence, T_1 is null homologous in V . Thus for $n \neq 1$, the map, induced by inclusion, of $H_{2n-1}(M)$ into $H_{2n-1}(V)$ is trivial. For $n = 1$, the kernel of this map includes the subgroup generated by T_1 . On the other hand, in the classical dimension, Casson and Gordon show that the kernel of the inclusion $H_1(M)$ into $H_1(V)$ is a subgroup of order the square root of the order of $H_1(M)$. As the order of the subgroup T_1 generates is of this order, we conclude that T_1 generates the kernel of this map.

It is not hard to see that any character which vanishes on the kernel extends to $H_1(V)$. The following duality argument shows that for n greater than one all characters on $H_1(M_q)$ to Q/Z extend to $H_1(V)$. Since the inclusion of $H_{2n-1}(M)$ into $H_{2n-1}(V)$ is 0, the boundary map of $H_{2n}(V, M)$ to $H_{2n-1}(M)$ is surjective. By duality, $H^2(V)$ maps onto $H^2(M)$. These groups are naturally isomorphic to $\text{Ext}(H_1(V))$ and $\text{Ext}(H_1(M))$ respectively, as M is a rational homology sphere and V is a rational homology ball. Finally, for finite groups G , $\text{Ext}(G)$ is canonically isomorphic to $\text{Hom}(G, Q/Z)$. This follows from the hom-ext exact sequence (see for instance [21]) for the coefficient sequence $Z \rightarrow Q \rightarrow Q/Z$. The argument in Casson and Gordon shows that for any χ of prime power order which extends to $H_1(V)$, $\tau(K, \chi) = 0$. $\square$

Calculating τ

In [1] it was shown that τ is related to the more easily computed σ and this was used as a means of proving that certain algebraically slice knots are not slice. That analysis could be imitated here. However, in the case of the examples at hand, it turns out that τ can be computed explicitly, yielding stronger results.

In order to describe τ we introduce certain Witt class invariants of knots, $w_{s/d}(K)$, which in turn contain signature invariants of K . Denote by N the $2n + 1$ manifold constructed as surgery on K , with 0-framing if $n = 1$. For each prime power d there is a unique homomorphism of $\pi_1(N)$ to Z_d which sends a positive meridian of K to the residue class of one. The manifold dN with this homomorphism bounds a $2n + 2$ manifold W over Z_d . We let $w_d(K)$ be $1/d$ times the difference of the Witt class of the intersection form on the twisted homology group $H_{n+1}(W, Q(\zeta_d))$, defined via the d -fold cover of W , and the intersection form on $H_{n+1}(W, Q)$. This class is viewed as an element in $L_e(Q(\zeta_d)) \otimes Z_{(2)}$, or $\otimes Q$ if d is even. The Galois group of $Q(\zeta_d)$ acts on the Witt group, and we let $w_{s/d}(K)$ denote the image of $w_d(K)$ under the Galois automorphism that carries ζ_d to $(\zeta_d)^s$. A standard argument shows that this $w_{s/d}$ is well defined. We note at this point that the inclusion of $Q(\zeta_d)$ into $Q(\zeta_d)(t)$ induces an injection of Witt groups [16] which remains injective under localization.

Signature functions are defined on these Witt groups, and we denote by $\bar{\sigma}_{s/d}(K)$ the signature of $w_{s/d}(K)$. In the Hermitian case the signature is defined by embedding $Q(\zeta_d)$ into C with ζ_d going to $e^{2\pi i/d}$ and taking the standard signature defined on complex Hermitian forms. In the skew Hermitian case the map given by multiplication by $(\zeta_d - \bar{\zeta}_d)$ induces an isomorphism from $L_2(Q(\zeta_d))$ to $L_0(Q(\zeta_d))$, the Hermitian group. Signatures are defined via this isomorphism. (Note that the isomorphism does not commute with the action of the Galois group.)

The value of $w_{s/d}(K)$ is determined by the Seifert form for K . A formula for $\bar{\sigma}_{s/d}(K)$ was presented in [22], but there are errors, one of which reappears in [24]. A derivation of formulas for w and $\bar{\sigma}$ will be presented in Appendix 2 at the end of the paper. For now the following statement suffices.

PROPOSITION 4. *If n is odd, $\bar{\sigma}_{s/d}(K) = \sigma_{s/d}(K)$. For n even, $\bar{\sigma}_{s/d}(K) = \sigma_{s/d}(K) + ((d - 2s)/d)\sigma(F)$, where F is a Seifert manifold for K .*

With w and $\bar{\sigma}$ now defined, we can proceed with our calculation of τ .

Let L' be the link formed when K is unknotted. This link is concordant to a boundary link (in fact it is strongly slice), and hence, by the proof of Theorem 3, τ of the relevant characters is zero. Let M'_0 be the manifold formed by surgery on the lift of K'_1 in the q -fold branched cover of S^{2n+1} branched over K'_1 . Let V' be a $(2n + 2)$ -manifold used to compute $\tau(K')$. We want to construct a manifold V which can be used to compute τ for the knot K_1 based on the manifold V' .

Note first that K_1 can be constructed from K'_1 by removing a neighborhood of a linking circle S to the left hand band in the Seifert manifold for K'_1 and replacing it with the complement of K . (To see this, observe that if a tubular neighborhood of S is removed from S^{2n+1} and replaced with the complement of K , then the resulting space is diffeomorphic to S^{2n+1} via a diffeomorphism which maps the complement of S to a neighborhood of K .) We want to lift this construction to the covers.

Suppose for the moment that surgery on K, N , bounds a $(2n+2)$ -manifold W over Z_{2m+1} . Let S' be the core of the surgery forming N . S lifts to q circles in M'_0 , say $\{S_i\}_{i=1} \dots q$. Attach q copies of W , say $\{W_i\}_{i=1} \dots q$, to V' identifying neighborhoods of S' with corresponding neighborhoods of S_i . Call the resulting manifold V . The boundary of V is our desired manifold, M_0 .

At this point V can be used to compute τ . If $n \neq 1$, the middle dimensional homology of V splits as a direct sum of that of V' and that of the W_i . Similarly, the middle dimensional homology of the relevant covers of V split into the sum of pieces coming from covers of V' and the W_i . Hence τ is also given as the sum of the contributions from each piece. In the classical dimension, the middle dimensional homology of V may no longer split as that of V' and the W_i . This can be corrected by picking W more carefully. We may factor the character on $H_1(N)$ through Z . As $\Omega_3(Z)$ is zero, we may pick W so that this character to Z extends. Then the homology splits as desired. As noted already, V' contributes 0 since L' is concordant to a boundary link, so we only need to compute the contribution from the W_i .

Under a representation of H_1 to $Z_d \times Z$ used to define τ , the generator of $H_1(N_i)$ goes to $(s_i, 0)$, for some s_i with $0 < s_i < d$. (Here N_i is the boundary of W_i .) Hence, the corresponding cover of W_i consists of disjoint copies of the d -fold cover of W_i freely permuted by the Z action. It follows that the value of τ is the sum of the $w_{s_i/d}(K)$. The exact values of s_i is given in Theorem 5 below.

The argument just given is easily modified if N does not bound over Z_d . As usual, for some r , rN does bound a W . In this case form the union of r copies of V' with W_1 and W_2 to construct V .

THEOREM 5. *If L_m is concordant to a boundary link, then for each prime power q and prime power d dividing $((m+1)^q - m^q)$, the sum $\sum_{i \in sS} w_{i/d}(J_1) = 0$, where S is the multiplicative subgroup generated by $m^*(m+1)$ in Z_d^* , the group of units in Z_d , and s is an arbitrary element in Z_d^* .*

Proof. As before, the complement of K_1 is formed from the complement of the knot formed with J_1 trivial by removing a tubular neighborhood of S and replacing it with the complement of J_1 . In the q -fold cover the neighborhood of each lift of S , S_i , is replaced with the complement of K . It follows that for a representation of $H_1(M_q)$ to Z_d (which vanishes on the T_i if $n = 1$), the value of τ is the sum of classes $w_{j/d}(J_1)$. Which values of j occur depends on the value of the representation on the S_i , as follows.

Suppose that S_1 goes to s in Z_d . Then, by Proposition 2, S_q goes to $s(m^*(m+1))$, S_{q-1} is mapped to $s(m^*(m+1))^2$ and so on. Letting $b = m^*(m+1)$, the sum of classes that arise is

$$\sum_{i=0}^{q-1} w_{(sb^i/d)}(J_1).$$

Modulo a , one has $mq = (m+1)q$. (Recall from Proposition 2, $a = ((m+1)^q - m^q)$.) Hence the order of b in Z_d^* divides q in the group of units in Z_d . It follows that the sum of classes is over the full coset of the multiplicative subgroup generated by b , with each term

occurring $q/(\text{order}(b))$ times. Hence, the product

$$(q/\text{order}(b)) \left(\sum_{i \in sS} w_{i/d}(J_1) \right) = 0.$$

It remains to eliminate the leading factor. If q is odd, then so is $(q/\text{order}(b))$, and as there is no odd torsion in the Witt group the result follows.

If q is a power of 2 we first consider the case n odd. Note then that b has even order, say $2k$, in Z_d . Let $x = b^k$. We have that $x^2 = 1$ in Z_d . Hence, either $x - 1 = 0$ in Z_d , $x + 1 = 0$ in Z_d , or both $x - 1$ and $x + 1$ are divisible by p (where $d = p^r$). This last case is impossible as p is odd. The case $x = 1$ also is ruled out. Hence, $x = -1$ in Z_d . It follows that terms in the sum $\sum_{i \in sS} w_{i/d}(J_1)$ occur in conjugate pairs.

By [5, 4.4, p. 50] Hermitian forms are determined by their rank, discriminant, and signatures. None of these are changed by conjugation, so one has that conjugate Hermitian forms are in fact equal in the Witt group. The only torsion in the Hermitian Witt group is 2 torsion, [5, Corollary 4.7, p. 51] so, as our class of interest is torsion and is 2 times another class, it is automatically 0 itself.

Finally, in the skew Hermitian case we have just seen that the classes in the sum occur in conjugate pairs. However, in L_2 conjugate classes are negatives of each other. (This is most easily seen by multiplying by $(\zeta_d - \bar{\zeta}_d)$ and working in the Hermitian setting.) Hence, the sum is again trivial. $\square$

The following corollary is immediate.

COROLLARY 5.1. *If L_m (in arbitrary dimension) is concordant to a boundary link and J_2 is unknotted, then for each prime power q and prime power d dividing $((m+1)^q - m^q)$, the sum $\sum_{i \in sS} \bar{\sigma}_{i/d}(J_1) = 0$, where S is the multiplicative subgroup generated by $m^*(m+1)$ in Z_d^* and s is an arbitrary element in Z_d^* .*

Letting $q = 2$, one has the following special case.

COROLLARY 5.2. *If n is odd and L_m is concordant to a boundary link, then $\bar{\sigma}_{s/d}(K) = 0$ for all $0 < s < d$, d a prime power divisor of $2m+1$.*

Proof. Initially, Corollary 5.1 gives only that $\bar{\sigma}_{s/d}(K) + \bar{\sigma}_{-s/d}(K) = 0$. However, as n is odd, these two terms are equal, and the result follows. $\square$

In the case that n is even, the signatures $\sigma_{s/d}(K)$ and $\sigma_{-s/d}(K)$ are negatives. This is the source of the mistake in [22] that was corrected and studied in [23] and [24]. The approach there, which works in this case as well is to consider the 3-fold cyclic branched cover instead of the 2-fold cover. For example in the case $m = 1$, $q = 3$, one has that $a = 7$ and $m^*(m+1) = 2$. The above result, along with Proposition 4 implies that $\sigma_{1/7}(J_1) + \sigma_{2/7}(J_1) + \sigma_{4/7}(J_1) + \sigma(F) = 0$. (See also [23, 24].)

Theorem 5 provides information about sums taken over cosets. These sums can be combined to yield simpler formula:

COROLLARY 5.3. *If d is a prime power divisor of $a = (m+1)^q - m^q$, then if L is concordant to a boundary link the sum $\sum_{i=1}^d \bar{\sigma}_{i/d}(J_1) = 0$.*

Proof. First assume that d is prime. As the sum over each coset is 0, it follows that the sum over the full multiplicative subgroup is also 0. Since d is prime, this gives the sum of all the d -signatures.

If $d = p^r$ for some r a similar argument shows that the sum of the d -signatures which are not (p^{r-1}) -signatures must vanish. By induction, the sum of the remaining signatures is also 0. $\square$

Remark. There are examples for which the Witt class invariant w_d provides stronger results than do signatures. For instance if $m = 1$ and J_1 is the 3 twisted double of the unknot then the signatures vanish (as the knot is of order two in the algebraic concordance group) but the obstruction described in Theorem 5 is nontrivial. These calculations will be presented in a separate paper.

§4. FURTHER APPLICATIONS OF DEGREE 2 CASSON-GORDON INVARIANTS IN DIMENSION 3

In [19] one particular collection of examples of links was used. The links were all of the form illustrated in Fig. 1, with J_2 unknotted. In this section we will present three new classes of examples of links which are not concordant to boundary links. In each case the examples will demonstrate the generality of situations where these techniques can be applied. Furthermore, each illustrates some new phenomena.

Note that for all of the examples in this section the Milnor $\bar{\mu}$ -invariants vanish. This follows from [3], noting that each example is a fusion of boundary links.

Example 1. Removing restrictions on K_2 . In the proof that L_m is not concordant to a boundary link which was presented in [19] it is easily seen that the only essential fact used about K_2 was that it has nontrivial algebraic linking number with the right hand band of the Seifert surface for K_1 , and trivial linking with the left hand band. For instance, if K_2 is replaced with its $(s, 1)$ cable the resulting link is not concordant to a boundary link.

This example is of interest, for if $2m + 1$ divides s , the complexity of (K_1, K_2) is 1. This complexity is defined in [4]. Hence, the explicit method used by Cochran and Orr in [4] which is based on 2-fold covers cannot be used to show that this link is not concordant to a boundary link. Of course their techniques are more general and by switching to higher degree covers those methods should also be able to show that these links are not concordant to boundary links.

Note that there are direct analogs of this example in all higher dimensions. These will be studied in Section 6.

Example 2. Removing the slice restriction on K_1 . If the knot J_2 is nontrivial the proof in [19] that L_m is not concordant to a boundary link fails. In this case K_1 may not be slice. We will now show that this restriction is unnecessary. Let J_2 be arbitrary.

THEOREM 6. *If L_m is concordant to a boundary link, then $\sigma_{s/d}(J_1) = 0$ for all prime power divisors d of $(2m + 1)$ and for all s , $0 < s < d$.*

Proof. If L_m is concordant to a boundary link then so is the link of three components built from K_1 and its mirror image, as illustrated in Fig. 2. (Remove a neighborhood of an arc joining the two ends of the concordance for K_1 and double the resulting manifold along

the $S^2 \times I$ to build the concordance.) Denote the component formed from K_1 and its mirror image by K'_1 , and the other two components by K'_2 and K'_2 .

Now K'_1 is slice, and in fact, K'_1 is slice in B^4 with the slice disk and the obvious Seifert surface for K'_1 , F'_1 , bounding a 3-manifold R disjoint from disjoint surfaces bounded by K'_2 and K'_2 in B^4 . Build this R in three steps. First form the 3-manifold R' bounded by the concordance for K_1 in the complement of the concordance of K_2 , as in [19]. Now R' can be doubled when the concordance is doubled. Finally, cap everything off in B^4 as in [19].

Let H be the half dimensional summand of $H_1(F'_1)$ which goes to torsion in $H_1(R)$. The Seifert form on $H_1(F'_1)$ vanishes on H , and this places strong restrictions on H . In fact the calculation of [18] shows that H must be one of the following, with respect to the basis $\{a, b, c, d\}$ formed by the cores of the bands taken from left to right in the diagram.

$$H_A = \langle (0, 1, 0, 0), (0, 0, 1, 0) \rangle$$

$$H_B = \langle (1, 0, 0, 0), (0, 0, 0, 1) \rangle$$

$$H_C = \langle (1, 0, 0, 0), (0, 0, 1, 0) \rangle$$

$$H_D = \langle (0, 1, 0, 0), (0, 0, 0, 1) \rangle$$

$$H_E = \langle (0, q, p, 0), (p, 0, 0, q) \rangle, p \text{ and } q \text{ relatively prime.}$$

However, as was used in the argument in [19], any element in H must link both K'_2 and K'_2 algebraically 0 times. That is, the projection of each element of H onto the first and fourth coordinates must be 0. This rules out all the possibilities except H_A .

The element $(0, 0, 1, 0)$ in H_A is represented by a curve with the knot type of J_1 . By the results of [10], J_1 must satisfy the given signature conditions. $\square$

Example 3. $\text{genus}(K_1) = 2$. The following example indicates how Casson–Gordon invariants can be used with two component links of higher genus. The particular choice is of interest in that the explicit methods of [4] fail to show that it is not concordant to a boundary link; the Blanchfield pairing of the associated covers of arbitrary degree will be seen to be Witt equivalent to those of a link which is concordant to a boundary link.

It should be pointed out that the calculations of [4] based on iterated covers are used there to detect invariants of a certain Γ group. Whether or not the links presented below map nontrivially to this Γ group is an open and fascinating question.

The link of interest is illustrated in Fig. 3. Let J_1 be the trefoil, J_2 be the inverse of the trefoil, and let $m = 1$. Again, if L is concordant to a boundary link, the Seifert surface F_1 for K_1 along with a slice disk bounds a 3-manifold R disjoint from a surface bounded by K_2 .

A calculation such as in [18] shows that there are five possible types for the kernel H . H_A through H_D are the same as in the above example, but H_E is replaced with

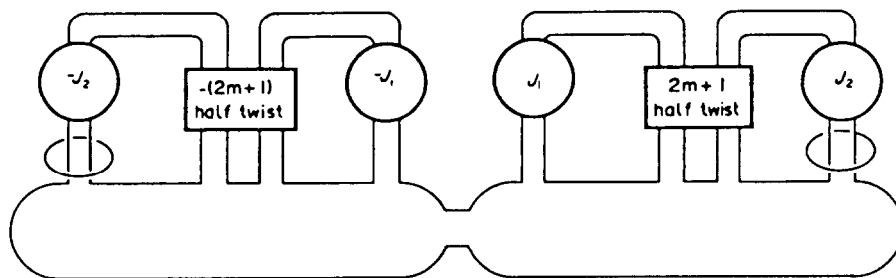


Fig. 2.

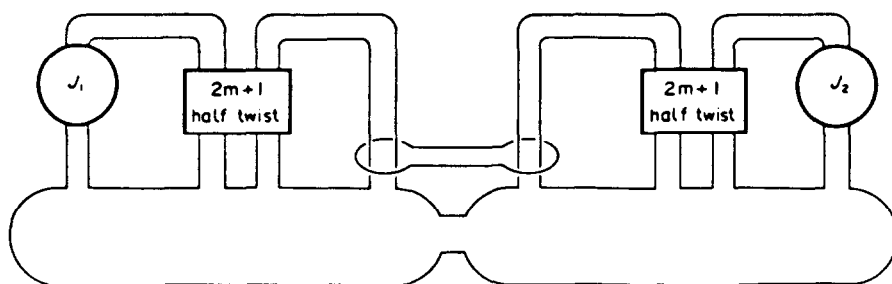


Fig. 3.

$\langle (q, 0, p, 0), (0, p, 0, -q) \rangle$, p and q relatively prime. As elements in H must link K_2 algebraically 0 times, the sum of the second and third components must be 0. This eliminates H_A , H_C , H_D , and H_E . Finally, H_B is ruled out by signature calculations.

Cochran and Orr define a class of invariants which arise from the Blanchfield pairing on specified covering spaces. These spaces are the composition of a finite cyclic branched cover over K_1 with some infinite cover the lifts of K_2 . Now if J_1 and J_2 were trivial, L would be concordant to a boundary link. The infinite cover is formed from the infinite cover arising with J_1 trivial by removing copies of $R \times B^2$ and replacing them with paired copies of the infinite cyclic covers of J_1 and $-J_1$. The sum of these added pairings is Witt trivial.

The previous examples all depended only on Casson-Gordon invariants arising from 2-fold covers. Using q -fold covers, where q is a prime power, yields further interesting families of examples. Information from many different covers can be combined to yield even stronger results.

As is usual, the methods of Casson and Gordon only give information about prime power signatures. Daryl Cooper has derived results corresponding to those we present with this restriction to prime powers removed. His methods do not apply in the higher dimensional setting.

Cooper's Theorem and consequences

In his thesis [7], Cooper proved a result related to Theorem 5 and its corollary without the restrictions concerning prime powers above. His work applies only in dimension 3. To state his result, first define an averaged signature function σ'_t by setting $\sigma'_t(K)$ to be the average of the one-sided limits of $\sigma_s(K)$, as s approaches t . Cooper proved the following:

THEOREM (Cooper). *If K is a slice knot in S^3 with a genus one Seifert surface F , then there is an essential curve J on F which is self annihilating with respect to the Seifert pairing. Moreover, J , when viewed as a knot, satisfies the signature condition of Corollary 5.1 for any d (prime power or not) relatively prime to m and $m + 1$, with σ' replacing $\bar{\sigma}$.*

An examination of the proof shows that J is characterized by: $[J]$ in $H_1(F, \mathbb{Q})$ is in the kernel of the map on rational homology induced by lifting F to the infinite cyclic cover of the 4-ball minus the slice disk. As this cover may be formed by cutting the 4-ball open along R and stacking and regluing an infinite number of copies of the complement, we see that Corollaries 5.1 and 5.2 hold in the classical case with the condition on d relaxed to " d is relatively prime to m and $m + 1$," and $\bar{\sigma}$ replaced by σ' .

In all dimensions these results place strong restrictions on the signature function for J_1 , and strong consequences result from these restrictions. However, the proofs depend on

careful number theory arguments which will not be presented here. In [8] it will be shown that if a signature function satisfies the criteria of Cooper's result with $m = 1$, then the only discontinuities are at roots of 1, and if the signature function is nontrivial, there are discontinuities at roots of -1 . This places strong restrictions on the factors of the Alexander polynomial.

Example 4. An infinite number of concordance classes. By changing the knot type of the components of the link L_m of Fig. 1 it is certainly possible to form new examples that are not concordant to the original ones. Here we will show that the simple generalizations of L_m discussed in Example 1 are (often) not concordant even though the individual components are. Fix $m = 1$, and let $L(s)$ be the link in Fig. 1 with J_2 trivial, but with K_2 replaced with its $(s, 1)$ cable. We will show that for an infinite collection of s the resulting links are not concordant.

First, a little number theory is called for. It was shown in Proposition 2 that the homology of the q -fold cover of S^3 branched over K_1 has first homology $Z_a \times Z_a$, where $a = (2^q - 1)$. Observe that if q and q' are relatively prime then so are a and a' . The argument is as follows: if $q > q'$ then

$$\begin{aligned}(2^q - 1, 2^{q'} - 1) &= ((2^q - 1) - 2^{q-q'}(2^{q'} - 1), 2^{q'} - 1) \\ &= (2^{q-q'} - 1, 2^{q'} - 1).\end{aligned}$$

Now apply induction. Form a set of primes, $\mathbf{P}$, by picking a prime divisor of $2^q - 1$ for each prime q .

THEOREM 7. *If s and t are distinct elements of $\mathbf{P}$ then $L(s)$ and $L(t)$ are not concordant.*

Proof. Suppose there is such a concordance with components C_1 and C_2 . As in [19] there is a 3-manifold R disjoint from C_2 with boundary the union of C_1 with two copies of the Seifert surface for K_1 , one at each end.

Remove the neighborhood of an arc on C_1 running from one end of the concordance to the other. The resulting space is B^4 with a link similar to that illustrated in Fig. 2 in its boundary. The only change is that K'_2 and K'_2 are replaced with their $(s, 1)$ and $(t, 1)$ cables. (J_2 can be taken to be trivial here.) Furthermore, K'_1 bounds a slice disk D for which there is a 3-manifold R with boundary $F'_1 \cup D$. The manifold R is disjoint from an annulus bounded by the two other components.

Suppose that s divides $2^q + 1 = a$, and form the q -fold cover, M_q . It is the connected sum of two copies of the q -fold cover of S^3 branched over K_1 . Hence $K_1(M_q) = (Z_a \times Z_a)^2$. The construction just given shows that the class $x = (s, 0, 0, t)$ is in the kernel of the inclusion of M_q into W , the branched cover of B^4 over the slice disk for K .

As we will be considering characters to Z_s , we now switch to Z_s coefficients and note that $H_1(M_q, Z_s) = (Z_s \times Z_s)^2$, and that $(0, 0, 0, 1)$ is in the kernel of the inclusion into W . As the linking form vanishes on the kernel, which is half dimensional, the kernel is generated by $(0, 0, 0, 1)$ and an element α which is of the form $(1, 0, 0, \varepsilon)$ or $(0, 1, 0, \varepsilon)$, where ε may or may not be 0.

To conclude the proof, note that the character consisting of projection on the third factor vanishes on the kernel, and hence extends over W . This character is trivial on one of the two summands. Now the additivity of τ [9, 16] can be applied, along with our previous calculations, to see that the value of τ on this character is given by an s signature of J_1 . The value of τ for powers of the character are given by other s signatures. If J is the trefoil, then some of these signatures will be nontrivial. $\square$

§5. AN ALGEBRAIC 3-DIMENSIONAL LINK CONCORDANCE GROUP

The work in this section applies only in dimension 3.

In analogy to the results in [10], we will give an algebraic definition of a group Ψ , and a map ψ which assigns to each two component link $L = (K_1, K_2)$ with 0 linking number an element in Ψ . We will also define a subgroup B of Ψ with the property that if L is concordant to a boundary link then $\psi(L)$ is in B . Finally, we will discuss algebraic means of determining if an element is in B . The reader is referred to [10] for background material.

Definition of Ψ . An element in Ψ is represented by a 4-tuple, $(V, \theta, \tau, \lambda)$, where V is a finitely generated Z module, θ is a bilinear pairing on V with $\theta - \theta^t$ nonsingular, τ is a function from A_p to W_p (as defined below) for all primes p , and λ is an element in V . Here A_p is determined by (V, θ) as follows: let $\varepsilon: V \rightarrow \text{Hom}(V, Z)$ be defined by $\varepsilon(x)(y) = \theta(x, y) + \theta(y, x)$ and let $A = \ker(\varepsilon) \oplus Q/Z$ and let A_p be the p -primary subgroup of A . (Note that $A_2 = 1$.)

W_p is defined to be the direct limit of the Witt groups of the field of rational functions of $Q(\zeta_p, r)$ tensored with $Z_{(2)}$. It is shown in [1] (see the version in *Topologie Perdue*) that the inclusion of $L_0(Q(\zeta_p, r)(t))$ into the direct limit is injective.

Addition is defined as in [10], with $(V_1, \theta_1, \tau_1, \lambda_1) \oplus (V_2, \theta_2, \tau_2, \lambda_2) = (V_1 \oplus V_2, \theta_1 \oplus \theta_2, \tau_1 \oplus \tau_2, \lambda_1 \oplus \lambda_2)$. An element $(V, \theta, \tau, \lambda)$ is called metabolic if there is a direct summand H of V such that: (1) $2 \text{ rank}(H) = \text{rank}(V)$, (2) $\theta(H \times H) = 0$, (3) $\tau(A_p \cap H \otimes Q/Z) = 0$, for all p , and (4) H is orthogonal to λ with respect to the form $\theta - \theta^t$. Define $-(V, \theta, \tau, \lambda) = (V, -\theta, -\tau, \lambda)$. The elements $(V_1, \theta_1, \tau_1, \lambda_1)$ and $(V_2, \theta_2, \tau_2, \lambda_2)$ are equivalent if $(V_1, \theta_1, \tau_1, \lambda_1) \oplus -(V_2, \theta_2, \tau_2, \lambda_2)$ is metabolic.

THEOREM 8. *The above relation defines an equivalence relation, and the direct sum operation defines a group operation on the set of equivalence classes.*

Proof. The argument here is essentially the same as that of the Cancellation Lemma of [10]. There is one additional feature. It must be shown that if $L \subset V_1 \oplus V_2$ is orthogonal to (x_1, x_2) and $K \subset V_2$ is orthogonal to x_2 , then $p(L \cap V_1 \oplus K)$ is orthogonal to x_1 . Here p is the projection onto V_1 . The proof is immediate. $\square$

Definition of ψ . Given a two component link (K_1, K_2) with linking number 0, we define $\psi(K_1, K_2) \in \Psi$ as follows. Let F_1 be a Seifert surface for K_1 in the complement of K_2 . Set $V = H_1(F_1)$, and let θ be the Seifert form for F_1 . The Casson-Gordon invariant τ is defined as in [10]. Finally, the map from $H_1(F_1)$ to Z given by $x \rightarrow \text{lk}(x, K_2)$ is a homomorphism and is hence given by the map $x \rightarrow x \cdot \lambda$ for some unique element λ in $H_1(F_1)$, where $x \cdot \lambda$ is the intersection number on F_1 .

THEOREM 9. *$\psi(K_1, K_2)$ is well defined, and depends only on the concordance class of (K_1, K_2) .*

Proof. Suppose that (K_1, K_2) is concordant to (K'_1, K'_2) via a concordance (C_1, C_2) . Let $K_1 = \partial F_1$ and $K'_1 = \partial F'_1$ as needed in the definition of ψ . The following argument proves both that ψ is well defined (take the trivial product concordance of a link to itself but use different surfaces F_1 and F'_1) and a concordance invariant. The argument is similar to that of Example 3.

The surface $F_1 \cup C_1 \cup F'_1$ bounds a 3-manifold R in the complement of C_2 . Removing an arc along C_1 yields a slice disk D for $K_1 \# K'_1$ in B^4 such that the Seifert surface $F_1 \cup F'_1$

union D bounds a 3-manifold R' in B^4 which is disjoint from an annulus bounded by K_2 and K'_2 . Let H be the kernel of the map from $H_1(F_1 \cup F'_1)$ to $H_1(R')/\text{torsion}$, induced by inclusion. $\square$

Detecting boundary links. It is certainly not true that boundary links are in the kernel of ψ . (For example, if (K_1, K_2) is in the kernel, then K_1 is algebraically slice.) However, concordance classes of boundary links can be detected via a subgroup of Ψ .

Definition. Let $\mathbf{B}$ be the set of elements in Ψ with representatives of the type $(V, \theta, \tau, 0)$.

THEOREM 10. *If (K_1, K_2) is concordant to a boundary link then $\psi(K_1, K_2)$ is in $\mathbf{B}$.*

Proof. Clearly, if (K_1, K_2) is boundary link, bounding the surface $F_1 \cup F_2$, then the linking function defined on $H_1(F_1)$ is 0, and hence $\lambda = 0$. $\square$

We now present two methods of showing an element is not in $\mathbf{B}$. First, let $f: \Psi \rightarrow \Gamma'$ be the forgetful map, where Γ' is defined as in [10], with the modification that τ now takes values in W_p . (f is given by $f((V, \theta, \tau, \lambda)) = (V, \theta, \tau)$.) Let $\mathbf{F}$ be the kernel of f .

THEOREM 11. $\mathbf{B} \cap \mathbf{F} = 0$.

Proof. The proof follows immediately from the definition. $\square$

Note that if K_1 is slice $\psi(K_1, K_2) \in \mathbf{F}$. The proofs that the basic examples of [19], and examples 1 and 3, are not concordant to boundary links could be rephrased in terms of this theorem. All that is shown is that $\psi(K_1, K_2) \neq 0$.

A general algebraic formulation of the proof that the links in Example 3 are not concordant to boundary links proceeds as follows. Define a group Ψ_2 as the set of 4-tuples, $(V, \theta, \tau, (\lambda_1, \lambda_2))$, where λ_1 and λ_2 are elements in V . Equivalence is defined as with Ψ , with the only change being that the metabolizer must be orthogonal to both λ_1 and λ_2 . The set of equivalence classes forms an abelian group as before.

There is a naturally defined "mirror" map, $\mathbf{m}: \Psi \rightarrow \Psi_2$ given by $\mathbf{m}((V, \theta, \tau, \lambda)) = (V \oplus V, \theta \oplus -\theta, \tau \oplus -\tau, ((\lambda, 0), (0, \lambda)))$. Let $\mathbf{M}$ be the kernel of $\mathbf{m}$.

THEOREM 12. $\mathbf{B} \subset \mathbf{M}$.

Proof. Pick a representative b of an element in $\mathbf{B}$ such that $\lambda = 0$. Let $H \subset V \oplus V$ be the set $\{(v, v)\}$. Clearly, H is a metabolizer for $\mathbf{m}(b)$. $\square$

Note that the proof in Example 2 can be restated in terms of this theorem. It was shown that $\psi(K_1, K_2)$ is not in $\mathbf{M}$.

Conclusion

We have seen that there is an abelian group $\Psi/\mathbf{B}$ and a map ψ' mapping 2-component links with linking number 0 to $\Psi/\mathbf{B}$. If a link is concordant to a boundary link it is in the kernel of ψ . The properties of $\Psi/\mathbf{B}$ remain to be studied.

§6. CASSON-GORDON INVARIANTS OF HIGH DIMENSIONAL LINKS

Let $L(K_1, K_2)$ be a link of two components in S^{2n+1} . Let M be the q -fold cover of S^{2n+1} branched over K_1 , where q is the prime power. Let $\tilde{K}$ be a component of the lift of K_2 to M .

As $\tilde{K}$ represents an element of $H_{2n-1}(M)$, by duality it determines an element in $\text{Hom}(H_1(M), Q/Z)$, χ . (See Section 2.) Since $\text{Hom}(H_1(M), Q/Z)$ is finite, it splits into its p -primary summands, for primes p . Let χ_p denote the projection of χ into the p summand.

Next, note that Z_q acts on the space M via covering translations. Hence, for any ρ in $Z[Z_q]$ the character $\rho\chi_p$ is defined. Note that $\tau(K_1, \rho\chi_p)$ is independent of the choice of $\tilde{K}$. This is because the covering transformations transitively permute the lifts of K_2 while fixing the lift of K_1 .

THEOREM 13. *For all p the value of $\tau(K_1, \rho\chi_p)$ is a link concordance invariant.*

Proof. Suppose that L is concordant to (K'_1, K'_2) . Let V be the manifold bounded by sM_0 used to compute τ . Let V' be the q -fold cyclic cover of the complement of neighborhood of the concordance from K_1 to K'_1 .

A simple geometric argument shows that $V \cup sV'$ has boundary M'_0 . Hence, $V \cup sV'$ can be used to compute $\tau(K'_1, \chi'_p)$. (The contributions coming from $F \times S^1$ and $F' \times S^1$ are the same since Seifert manifolds for K_1 and K'_1 are cobordant; recall that the punctured manifolds union the concordance bounds the manifold R in $S^{2n+1} \times I$.)

The representation of V' to Z_p is determined by the relative homology class represent by the lift of the concordance of K_2 . By Casson and Gordon's original argument in [1], $H_*(, Q(\zeta_d)(t))$ vanishes for V' and it also vanishes on the q -fold cyclic cover of the complement of K . Hence, the invariants are the same. $\square$

THEOREM 14. *If L is concordant to a boundary link, then $\tau(K_1, \rho\chi_p) = \tau(K_1, 0)$.*

Proof. By Theorem 13, it is enough to check this for boundary links. For a boundary link, the lift of K_2 is null homologous in the cover, and hence it determines a trivial representation. $\square$

Remark. It is not difficult to show that if K_1 is algebraically slice then $\tau(K_1, 0) = 0$. One uses Litherland's approach to the algebraic knot concordance group. See [16, Sec. 2] for details.

Application. Let $L^{2n+1}(s)$, $s \in \mathbf{P}$, now denote the higher dimensional analogues of the links in S^{2n+1} described in Example 4. We assume now that n is odd, and let J_1 be a knot having the signature function of the trefoil. The following theorem corresponds to Theorem 7.

THEOREM 15. *None of the links $L(s)$ are concordant to boundary links. For s and t distinct primes in $\mathbf{P}$, $L(s)$ and $L(t)$ are not concordant.*

Proof. For the first statement, let $t \in \mathbf{P}$ be a prime distinct from s . Consider the q -fold branched cover M , branched over the first component of $L(s)$ where t divides $2^q - 1$. With Z_t coefficients, $H_{2n+1}(M) = Z_t$, and the lift of K_2 is s times a generator, and, since s and t are distinct, is a generator itself. Hence, the character it determines from $H_1(M)$ to Z_t is nontrivial. The proof of Theorem 5 and its corollaries now go through to show that τ is given by a sum of signatures of K . By our choice of signature function for J_1 these sums will not be zero.

To see that $L(s)$ and $L(t)$ are not concordant, consider also the d -fold branched cover M' over the first component of $L(t)$. Since K_2 is divisible by t , its lift to the cover is trivial with Z_t coefficients. Hence, at this end we are considering $\tau(M', 0)$ which equals 0. $\square$

Comment. In dimension $2n + 1$ with n even there is an added difficulty. Here, the signature function satisfies $\sigma_a(K) = -\sigma_{-a}(K)$ for knots K . Hence, the sum over all t -signatures is 0. Suppose though that J_1 has the signature function $\sigma_a(J_1)$ which is the same as that for the trefoil for $\text{Im}(x) > 0$. We can apply the full strength of Theorem 5 to get results.

For instance, if $2q - 1$ is prime the subgroup generated by 2 is rather small, and it is easily verified that the sum given by Theorem 5 is nontrivial. Unfortunately, it is unknown whether or not there are an infinite number of such primes. All that is actually needed to produce an infinite number of examples are an infinite collection of numbers of the form $2^q - 1$ with q prime such that each one has prime factors which are large compared to q . We have not completed this number theory.

Generalizations to several component links. In the case that L has more than two components, say $L = (K_0, K_1, \dots, K_n)$, each K_i determines a character χ_i on M , the q -fold cover of S^{2n+1} branched over K_0 . (We now drop the p from the notation.) Theorem 14 remains true with $\rho\chi$ replaced by any linear combination of the χ_i with coefficients in $\mathbb{Z}[\mathbb{Z}_q]$. The arguments are as above.

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APPENDIX 1

Here we prove Proposition 2. It is split into two parts to clarify the exposition.

PROPOSITION 2a. *If $n = 1$, $H_1(M_q) = Z_a \times Z_a$, where $a = ((m+1)^q - m^q)$. If $n > 1$, $H_1(M_q) = Z_a$, $H_{2n-1}(M_q) = Z_a$, and all other homology groups are trivial other than in dimensions 0 and $2n+1$. In $H_1(M_q)$, $S_i = m^*(m+1)S_{i+1}$, $i < q$, and $S_q = m^*(m+1)S_1$, where m^* is the multiplicative inverse of $m \bmod(a)$.*

Proof. The proof is based on Seifert's algorithm, as presented in [20], using a Mayer-Vietoris argument. We present the case $n = 1$.

Denote the linking circles to the two bands in the Seifert by S and T . Recall that their lifts are denoted $\{S_i\}_{i=1,\dots,q}$ and $\{T_i\}_{i=1,\dots,q}$, with each set cyclically permuted by the Z_q action.

One concludes from the Mayer-Vietoris argument in [20] that the S_i 's and T_i 's generate $H_1(M_q)$. The relations are given by $m(S_i) = (m+1)(S_{i+1})$ for $i = 1, \dots, q-1$, and $mS_q = (m+1)(S_1)$ in $H_1(M_q)$. A similar set of relations hold for the T_i 's, with the role of m and $m+1$ reversed.

Multiplying the relation $m(S_1) = (m+1)(S_2)$ by m yields $m^2(S_1) = (m+1)^2(S_3)$. Multiplying by m another $q-2$ times yields the relation $m^q(S_1) = (m+1)^q(S_1)$, or $((m+1)^q - m^q)(S_1) = 0$. Since m is relatively prime to the order of S_1 there is some integer m^* such that $m^*mS_1 = S_1$. Multiplying the relation $m(S_1) = (m+1)(S_2)$ by m^* yields that $S_1 = m^*(m+1)S_2$. By symmetry, $S_i = m^*(m+1)S_{i+1}$, $i < q$, and $S_q = m^*(m+1)S_1$. Via repeated substitutions one finds that S_1 generates $H_1(M_q)$ subject to the relation $S_1 = (m^*(m+1)^q)S_1$. Finally, multiplying both sides by m^q yields the desired result.

A similar argument works for the T_i 's.

In higher dimensions the S_i 's generate homology in dimension 1, the T_i 's in dimension $2n-1$. The rest of the higher dimensional calculation follows as above. $\square$

PROPOSITION 2b. *For $n = 1$, the (nonsingular) linking form on $H_1(M_q)$ with values in Q/Z is hyperbolic with respect to the basis of $H_1(M_q)$ represented by S_1 and T_1 . In dimensions greater than 3 the linking form pairs these generators of H_1 and H_{2n-1} nontrivially.*

Proof. First, in dimension 3. Let γ be a curve representing the core of the right band on F , and let $i_+(\gamma)$ and $i_-(\gamma)$ represent the positive and negative pushoffs of γ from F . Then $i_+(\gamma)$ and $m(S)$ form the boundary of a chain, C_+ , in $S^3 - F$. Similarly, $i_-(\gamma)$ and $(m+1)(S)$ form the boundary of a chain, C_- , in $S^3 - F$. These chains can be lifted to the cover and pieced together to form a chain in M_q with boundary a multiple of S . (Just imitate the construction of the relations in M_q .) This chain in the cover can in turn be used to compute linking numbers.

The result in higher dimensions follows from the fact that the linking form is nontrivial. $\square$

APPENDIX 2

In Section 2 we defined an invariant $w_d(K)$ in $L_d(Q(\zeta_d) \otimes Z_{(2)})$, or $\otimes Q$ if d is even. We wish to give a formula for $w_d(K)$ in terms of a Seifert matrix A for K .

For a square matrix M and a complex number α of norm one, let M_α denote $(1-\alpha)M + (-1)^{n+1}(1-\bar{\alpha})M^*$, where $*$ denotes conjugate transpose. Throughout this appendix we

will fix an embedding of $Q(\zeta_d)$ into C and let ζ stand for $e^{(2\pi i/d)}$. Let Δ denote the diagonal form with entries $(1 - \zeta), (1 - \zeta^2), \dots, (1 - \zeta^{(d-2)})$. We will let Δ_ζ be the skew Hermitian form $(1 - \zeta)\Delta - (1 - \bar{\zeta})\Delta^*$. Note that Δ_ζ has rank $d - 2$. Denote by $[x]$ the Witt class of x . We will use Kauffman's conventions [13] regarding linking numbers and Seifert pairings. For instance, by Lemma (2.1) of [13] the signature of the Seifert manifold, F , is given by minus the signature of the Seifert matrix symmetrized when n is even. Finally, we use Sign as shorthand for signature.

THEOREM 16. *If A is a Seifert matrix for K we have $w_d(K) = [A_\zeta]$ when n is odd, and $w_d(K) = [A_\zeta] + (1/d) (\text{Sign}(A + A')[\Delta_\zeta])$ when n is even.*

Proof. The discussion on pages 186–187 of [1] shows that the twisted homology group of a d -fold covering space with $Q(\zeta)$ coefficients is the same as the ζ -eigenspace of the generating covering translation on the ordinary homology with $Q(\zeta)$ coefficients. Moreover, the equivariant intersection pairing differs from the ordinary intersection pairing extended to a Hermitian form and then restricted to this eigenspace only by a factor of d , which is a norm of $Q(\zeta)$ over $Q(\zeta + \bar{\zeta})$. Thus the two forms are Witt equivalent.

Suppose that F is a Seifert manifold for K with Seifert matrix A . Consider the d -fold branched cover of D^{2n+2} along F , F pushed into D^{2n+2} . The intersection form on the ζ -eigenspace is given by A_ζ . (See [13], (5.7), and [9], (3.4).) If we add a 2-handle to D^{2n+2} along K (with 0-framing if $n = 1$) to form W_1 , then the boundary of W_1 is the N described in the definition of w_d . Let $\bar{F}$ denote F' union the core of the 2-handle. Note that the branched cover of D^{2n+2} along F' extends to a branched cover of W_1 along $\bar{F}$. Let W_2 denote W_1 with an open tubular neighborhood of $\bar{F}$ removed. The intersection form on the unbranched cover of W_2 is also given by A_ζ as adding the handle and deleting the neighborhood does not affect the ζ -eigenspace.

Notice that the boundary of W_2 is N and $-\bar{F} \times S^1$ where the cover is trivial on the $\bar{F}$ factor and the covering translation is given by a rotation of $2\pi/d$ on the S^1 factor.

Consider the d -fold branched cover of S^2 along d points, B . Let B' denote B with equivariant open neighborhoods of the fixed points deleted. Then $\bar{F} \times B'$ provides a null bordism of d times the cover of $-\bar{F} \times S^1$. The intersection form on the ζ -eigenspace of B is Witt equivalent to the one given by $-\Delta_\zeta$. (See the proof of (5.2) in [9], but note that the form there has a single zero on the diagonal which we have deleted to mod out by the radical.) By the proof that the ordinary signature is multiplicative, the form on $\bar{F} \times B$ is Witt equivalent to $-\text{Sign}(\bar{F})[\Delta_\zeta]$ if n is even and is Witt trivial if n is odd. The form on $\bar{F} \times B'$ is the same.

Gluing together d copies of the cover of W_2 and one copy of $\bar{F} \times B'$, we obtain a nullbordism of the cover of N . As the cover of $\bar{F} \times S^1$ has a trivial ζ -eigenspace, the intersection form is the direct sum of the forms on the pieces. Note that downstairs we have a manifold with Witt trivial intersection form over Q . Recalling that $\text{Sign}(\bar{F})$ is $-\text{Sign}(A + A')$ completes the proof. $\square$

To fix our conventions we make the following definitions.

Definition. Let $\sigma_{s/d}(K) = \text{Sign}(A_{\zeta^s})$ if n is odd, and set $\sigma_{s/d}(K) = \text{Sign}((\zeta - \bar{\zeta})A_{\zeta^s})$ if n is even. Set $\delta_{s/d}(K) = \text{dis}[A_{\zeta^s}]$.

The discriminant, dis , is defined in [5, p. 43] for Hermitian forms. It takes values in the multiplicative group of units in $Q(\zeta + \bar{\zeta})$ modulo norms from $Q(\zeta)$. For a skew Hermitian form, w , the discriminant is defined to be the discriminant of the Hermitianization, $(\zeta - \bar{\zeta})w$.

COROLLARY 16.1. $\bar{\sigma}_{s/d}(K) = \sigma_{s/d}(K)$ if n is odd, and $\bar{\sigma}_{s/d}(K) = \sigma_{s/d}(K) - (d - 2s)/d \text{Sign}(A + A')$ if n is even. Furthermore, $\text{dis}[w_{s/d}(K)] = \delta_{s/d}(K)$.

Proof. The signature of the form Δ_{ζ^s} is $d - 2s$. This is proved in Proposition (5.2) of [9]. The form Δ does not enter into the discriminant formula as $A + A'$ is even, unimodular, and integral, and so has signature divisible by eight. $\square$